

Bounds and intervals around nonzero cylinder powers in symmetric dioptric power space

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Abstract. We seek to analyze the geometry and explain how bounds and intervals of nonzero purely cylindrical powers are obtained and applied in symmetric dioptric power space and envisaged in the clinic. The principal powers at zero and at the focus at the cylinder power of a lens are subject to the same uncertainty when measured. Accompanying these uncertainties is an error in axis position. Error cells are constructed for typical cylinder axes and an associated power. The geometry contains an elegant clinical determination for cross-cylinder compensation of astigmatism in terms of calculation friendly quantities. The extreme positions in the error cells define bounds for the cross-cylinder powers and their meridians. When clinical powers in a chosen error cell are transposed, the new powers are within a different cell. This ambiguous cell pair maps to a single cell in an antistigmatic plane around cross-cylinder powers. © 2009 Society of Photo-Optical Instrumentation Engineers. [DOI: 10.1117/1.3079809]

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1 Introduction

Like most measurements, the astigmatic compensation of a patient's vision has indeterminate numbers estimated for it. Bounds in the vicinity of this measurement and intervals are believed to contain the patient's "true" or "exact" demand. A theoretical concept like uncertainty is made tangible and substantial by well-known, visible, plane figures from coordinate geometry that allow us to identify extreme values for bounds of antistigmatic coordinates, and are the best way to analyze or process intervals. The figures beautifully reveal a clinical process for compensation of astigmatism in calculation friendly quantities.

An error cell is a set of powers in the vicinity about a prescription that images within an estimated diopter range of the retina. This condition defines the shape and size of the cell.¹ Clinical powers are transformed to calculation friendly quantities by equations that have an elementary geometric interpretation. The error cells around nonzero purely cylindrical powers become cells in symmetric dioptric power space.

Recent works^{2,3} have challenged the accepted view of how subjective refraction is performed, and offer a radically different routine in a plane. A unique point in power space represents every discrete symmetric power (the ordinary astigmatic power familiar to optometry). To determine the antistigmatic component of the refractive compensation, the routine needs Jackson cross-cylinders only, that is, lenses in a range of purely antistigmatic powers.

Powers in error cells in terms of cylinder power F_C and axis A are converted to be antistigmatic powers in unique cells in a plane of symmetric dioptric power space. The plane, where the coordinates plotted along the F_J and F_K axes have identical units, houses those powers where dioptric power ma-

trices have zero trace, and is perpendicular to planes previously considered.¹ The F_J axis contains all cross-cylinder powers whose principal meridians are vertical and horizontal. The F_K axis represents cross-cylinder powers whose principal meridians are at 45 and 135 deg. The powers have zero spherical equivalent or are Jackson cross-cylinders.

A unique error cell with radial and tangential dimensions in an antistigmatic plane is the image of a pair of rectangular clinical error cells. Bounds of the individual antistigmatic coordinates of a power matrix and intervals are obtained for discrete powers of cylinder lenses and their axes. The bounds give the extreme positions of the principal meridians of a cross-cylinder lens and the extreme values of its principal powers. The patient's exact compensation is believed to be between these extremes. We model the Jackson cross-cylinder procedure for the unambiguous powers in an error cell, and thus include the uncertainty that accompanies the measurements. Bounds of powers and axes of cylindrical lenses and intervals around nonzero power as spherocylindrical powers in symmetric power space are examined independently.

It is important in research to include error and its effect as well as account for patterns in measurement data^{4,5} that are not the result of transformations. Cylindrical lenses of zero power produce different intervals around powers and axes and are not discussed.

It is verified¹ that error cells plotted with sphere and cylinder powers and axis in positive cylinder form are not the same as cells where powers are transposed to negative cylinder form. This noninvariance of cells does not carry across to the new cells when they are transformed to symmetric dioptric power space. Invariance under spherocylindrical transpo-

sition is a necessary requirement of all meaningful ophthalmic parameters. Thus the bounds and intervals in symmetric dioptric power space are truly meaningful. Hence, for quantitative analyses, the traditional clinical spherocylindrical representation of power must be replaced by the invariant symmetric dioptric power space. In fact, the invariance one observes in symmetric power space is a feature of dioptric power space itself and is independent of the noninvariance from the world of spherocylindrical power.

In symmetric dioptric power space, the axes of the cylinders distinguish the planes containing the axis of scalar powers from each other. The invariance exists for all axes, and thus for all planes and the entire symmetric dioptric power space. Lattice points subdivide the cylinder-sphere plane and any plane containing the axis of scalar powers in symmetric power space into error cells. The area of the surface of an error cell in a plane of the axis of scalar powers is half of the area of the cell about the same power in the cylinder-sphere plane, valid for both spherical and astigmatic powers.

Further,⁶ symmetric intervals estimated for powers in sphere, cylinder, and axis lead to parallelepipedal error cells. Every power on or in the cell is transformed into a typical power on or in cells in symmetric power space. Coordinates of the power matrix are obtained using transformations for discrete powers. The error cell surrounding sphere and cylinder powers and axis is converted to a cell about scalar, orthogonal, and oblique antistigmatic powers. In turn, the cells are converted to cells around entries of the dioptric power matrix, principal powers, and principal meridians.

The error cells are projected onto the planes of cylinder and sphere and cylinder and axis. Error cells in the cylinder-axis plane image onto an antistigmatic plane in symmetric dioptric power space. The powers give rise to a family of arcs centered at the origin and a family of lines through the origin.

When restricted to thin systems with perpendicularly crossed toric faces, the representation^{7,8} reduces to something similar to that of Thibos, Wheeler, and Horner;⁹ matrix $F_J \mathbf{J}$ corresponds to their scalar J_0 and matrix $F_K \mathbf{K}$ to their scalar J_{45} . Matrix notation holds for the power of all systems, thin or with obliquely crossed toric faces, thick and with perpendicularly crossed toric faces, so that it better reflects the true character of dioptric power, that of a matrix. Furthermore, it is not confined to power but holds also for all the fundamental properties and many derived properties of linear optics.

2 Constructing Bounds and Intervals

Suppose a reported power of, for example, 1.25 D of a cylindrical lens follows from a bracketing and optimization process that converges with the patient's responses to "which is better, 1 or 2, or are they the same?" and the practitioner's judgement. The practitioner brings bounds as close together as possible. The reported power was obtained by the methods of classical texts.^{10,11} The bounds (1 and 2) seldom coincide, so that some small region remains after the new compensation has been dispensed to the patient. In our example some hypothesized "true" figure or patient's "exact" demand has been rounded to the power 1.25 D¹² that a supplier typically has on the shelf or that a measuring instrument or routine is capable of reporting. An interval must be an indeterminate quantity, or an "uncertainty," for if one knew what the bounds were pre-

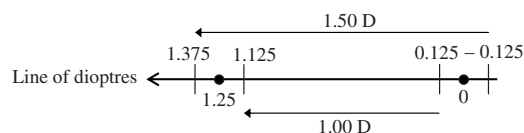


Fig. 1 A line of dioptres. The hypothetical power is estimated to be within 0.125 D of the rounded number, that is, plano power is between -0.125 and 0.125 D, and reported power 1.25 D is between 1.125 and 1.375 D. Then the true value of the cylinder lens power, reported as 1.25 D, is estimated to be between 1.00 and 1.50 D.

cisely, the "true" or "exact" patient demands could be adjusted for, and estimates for bounds and mathematics would be unnecessary.

A pure thin cylindrical lens like any spherocylindrical lens has two principal powers. The cylinder lens brings incident rays parallel to the line of diopters to focus in a line parallel to the axis of the cylinder at this nonzero power. To complete the picture, imagine the other principal power at zero to focus on a line of diopters at zero perpendicular to the focal line of the nonzero lens. These focal positions define the length of an interval in diopter that is the power of the cylinder lens.

Let us estimate that the hypothetical power of a cylinder lens lies within 0.125 D of the rounded value. Thus 0 D power is between -0.125 and 0.125 D, and similarly, 1.25 D cylinder power is between 1.125 and 1.375 D (see Fig. 1) as the conjugate nature of principal foci of spherocylindrical lenses demand.¹ Then the true value of the cylinder lens power, reported as 1.25 D, is estimated to be between 1.00 and 1.50 D (1 and 2), as shown in Fig. 1. Similarly the "true" or "exact" axis angle of the lens is estimated to be in an interval. The error in the principal powers is 0.125 D and the error in the power of the cylinder lens is 0.25 D. Thus principal powers are 0 ± 0.125 D and 1.25 ± 0.125 D, and the power of the cylinder lens is 1.25 ± 0.25 D. An experienced researcher has the ability to understand, quantify, and communicate uncertainty in measurements. A measurement that does not contain the likely interval of possible errors has limited information. The true value differs surely from the one that is reported, and therefore one has no idea of the possible size of this difference.

The region around power 0 D could equally be around a spherical lens. Although the exact hypothetical power is regarded as cylindrical, in most cases the power does contain some sphere, and therefore, is actually astigmatic. It is known that during refraction, changes in sphere power give rise to adaptations in sphere power to keep the circle of least confusion on the retina. In symmetric dioptric power space, this preserves the independence of the scalar power from an antistigmatic power in the determination of the power of cross-cylinder compensation. Then the trace of the dioptric power matrix remains invariant.¹ In Fig. 1 the arrows 1.00 D and 1.50 D represent positive powers of cylinder lenses. By reversing the arrows, the cylinder lens power is estimated to be between -1.50 D and -1.00 D. Let a line of axes similar to that of Fig. 1 illustrate bounds in axis angles.

Previous work⁶ shows that no common independent variables exist among proper values and vectors of the power matrices for spherocylindrical powers, so that proper values and vectors of the dioptric power matrix are functionally in-

Table 1 Rectangular coordinates of the corner points of the error cell for the reported cylindrical powers 1.25 D and -1.25 D, whose axes are at 0 and 90 deg, as shown in Fig. 2.

	Axis (deg)	Axis (deg)	+Cyl (D)	-Cyl (D)
E ₂	-2.5	87.5	1.50	-1.50
E ₃	-2.5	87.5	1.00	-1.00
E ₂	2.5	92.5	1.50	-1.50
E ₃	2.5	92.5	1.00	-1.00

dependent. Thus an interval of cylinder powers does not determine an interval of corresponding cylinder axes or vice versa.

The best way to represent axes and corresponding powers of the cylinder lens is in the plane with a rectangular set of axes.⁶ Any hypothetical cylindrical component within the interval of the rounded power has a corresponding hypothetical axis located anywhere within the axis interval. Intervals of axis angles and corresponding cylinder power intervals combine to form error cells around each reported cylinder power.

3 Illustrating Examples

For the reported cylindrical powers 1.25 D and -1.25 D whose axes are at 0 and 90 deg, as shown in Fig. 2, powers at the corners of the error cells are listed in Table 1. The axes are estimated to range from -2.5 to 2.5 deg and 87.5 to 92.5 deg.

The cylinder powers $F_C = \pm 1.25$ D with corresponding axis at $A=0$ are the points Q and P plotted on the axis of cylinder powers F_C in Fig. 2. The cell about Q (with positive power) images on the cell about P and both straddle the cylinder axis at $A=0$.

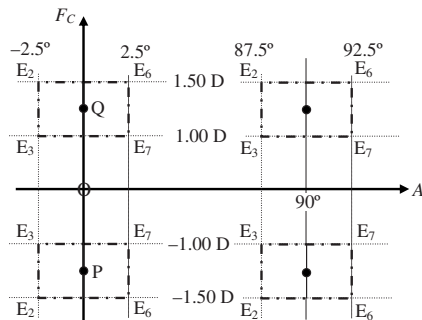


Fig. 2 Error cells in positive and their images in negative cylinder power at a common axis in an axis-cylinder plane about discrete powers are seen with their spherocylindrical transposes. Imagine a third axis for spherical power (not shown) perpendicular to the paper. The cells are used to study the effects of transposition and change of sign of cylinder power only on cross-cylinder powers and two typical cells that surround them. The figure is schematic and shows rectangular surfaces of a parallelepiped (no scale is implied). Power of the cylinder lens is plotted in diopter along the F_C axis and the corresponding axis of the cylinder lens is plotted in degrees along the A axis.

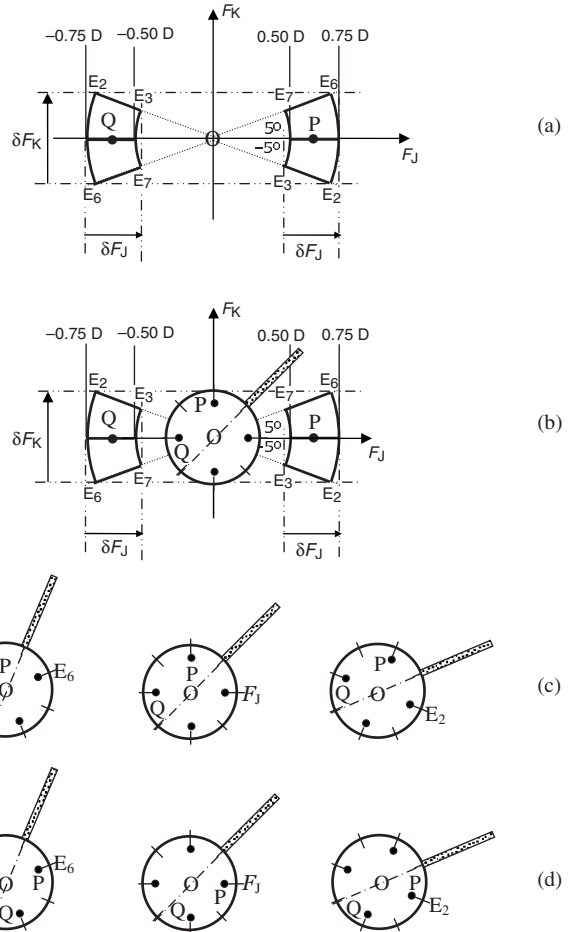


Fig. 3 Error cells that are images of all the cells from Fig. 2 are bounded by two radial lines E_2E_3 and E_6E_7 , and the arcs E_2E_6 and E_3E_7 of the same circles on either side of the F_K axis. Drawings are not to any scale and the points are a plot of rows in Table 1 that satisfy Eqs. (1). The F_J value is found in the interval whose lengths are denoted by δF_J . The F_K value is in the interval between E_2 and E_6 . The length of this interval is denoted by δF_K . Cross-cylinder meridians are seen rotated through 10 deg, and lenses are flipped so that positively powered meridians OQ are replaced by negatively powered meridians OP or vice versa. The arcs E_2E_6 and E_3E_7 of the same circles on either side of the F_K axis represent extreme powers of positive and negative cross-cylinder lenses. Although we usually work with cylinder lenses as compensation and a cross-cylinder lens as a probe, this method uses the cross-cylinder lenses as both probe and compensation.

The error cell about -1.25×90 is the spherocylindrical transpose of the cell about 1.25×0 ; similarly, the cell about 1.25×90 is the transpose of the lower cell about -1.25×0 . Because of the invariance under transposition, every vertical error cell pair related as shown in Fig. 2 forms the single diametrically opposite cell pair shown in Fig. 3(a). An error cell about the power and axis at Q is on the left in Fig. 3(a).

By transforming reported powers⁸ of cylinder 1.25×0 and -1.25×0 (at Q and P in Fig. 2), the rectangular antistigmatic coordinates F_J and F_K of the dioptric power matrix are obtained as follows:

$$\begin{pmatrix} F_J \\ F_K \end{pmatrix} = -\frac{1.25}{2} \begin{pmatrix} \cos 2(0) \\ \sin 2(0) \end{pmatrix} = \frac{5}{8} \begin{pmatrix} -1 \\ 0 \end{pmatrix} D \text{ and}$$

$$\begin{pmatrix} F_J \\ F_K \end{pmatrix} = -\frac{1.25}{2} \begin{pmatrix} \cos 2(0) \\ \sin 2(0) \end{pmatrix} = \frac{5}{8} \begin{pmatrix} 1 \\ 0 \end{pmatrix} D, \quad (1)$$

respectively. Equations (1) show the coordinates $\begin{pmatrix} F_J \\ F_K \end{pmatrix}$ at points Q and P on opposite sides of the origin O in Fig. 3(a). Polar coordinates $(r; \theta) = (1.25/2; 0)$ that identify point P emanate from axis and cylinder $(0; -1.25)$. The polar coordinates of Q are then $(-5/8; 0) = (5/8; 180)$. A circle with diameter PQ and with equation $F_J^2 + F_K^2 = (5/8)^2$ could be drawn. The diameter PQ with equation $F_K = 0$ then intersects the circle at $F_J = \pm 5/8 D$.

The antistigmatic coordinates for each of the cylinder powers 1.25×90 (transpose at P) and -1.25×90 (transpose at Q) are

$$\begin{pmatrix} F_J \\ F_K \end{pmatrix} = -\frac{1}{2}(1.25) \begin{pmatrix} \cos 180 \\ \sin 180 \end{pmatrix} = \frac{5}{8} \begin{pmatrix} 1 \\ 0 \end{pmatrix} D \text{ and}$$

$$\begin{pmatrix} F_J \\ F_K \end{pmatrix} = -\frac{1}{2}(-1.25) \begin{pmatrix} \cos 180 \\ \sin 180 \end{pmatrix} = \frac{5}{8} \begin{pmatrix} -1 \\ 0 \end{pmatrix} D.$$

Cylinder powers that are spherocylindrical transposes are seen to have the same antistigmatic coordinates. Clearly the coordinates are invariant under transposition. This is valid for all powers. At corresponding points in any diametrically opposite error cells arcs of the same circle and a line intersect. The radii of the arcs represent the powers of cross-cylinders of opposite sign and the direction of the line (or a line at 45 deg to the diameter) represents the direction of the lens. Thus error cells in an antistigmatic plane are a set of points defined by a discrete power, where cross-cylinder lens meridians and their powers intersect. In Fig. 3 one sees the handles at positions along a line at 45 deg to the diameters as the practitioner flips a cross-cylinder and presents powers of opposite sign to the patient.

A line emanating from the page in the vicinity of O in Fig. 3 may be regarded as the line of sight of a patient's eye. All of the error cells from Fig. 2 are imaged without a superposed cross-cylinder lens in Fig. 3(a) and with a superposed cross-cylinder lens in Fig. 3(b). In Fig. 3(b) the cross-cylinder meridian OQ of positive power is aligned with OQ in Fig. 3(a). The letter O and the line OE₆ of the cross-cylinder on the left in Fig. 3(c) must be brought into coincidence with those in Fig. 3(a). Similarly, the cross-cylinders on the right in Fig. 3(c) have OE₂ coincide with that of Fig. 3(a). The largest and smallest radii of arcs of a family of concentric circles in an antistigmatic plane in Fig. 3(a) represent powers of cross-cylinder lenses ranging from -0.75 to $-0.50 D$ on one side of the origin and 0.50 to $0.75 D$ on the opposite side.

A family of diameters in Fig. 3(a) sloping from the angle at -5 to 5 deg represents the principal meridians of cross-cylinder lens powers. The diameters with extreme slopes intersect the circles with extreme radii and locate error cells between line segments and arcs of circles. The mathematics that give rise to Fig. 3(a) depict 1. the clockwise rotation of the cross-cylinder handles of the lenses through 10 deg in the Fig. 3(c), and 2. the flipping of the lenses so that positive meridians in Fig. 3(c) are replaced by negative meridians in Fig. 3(d) or vice versa. Although we usually work with cylin-

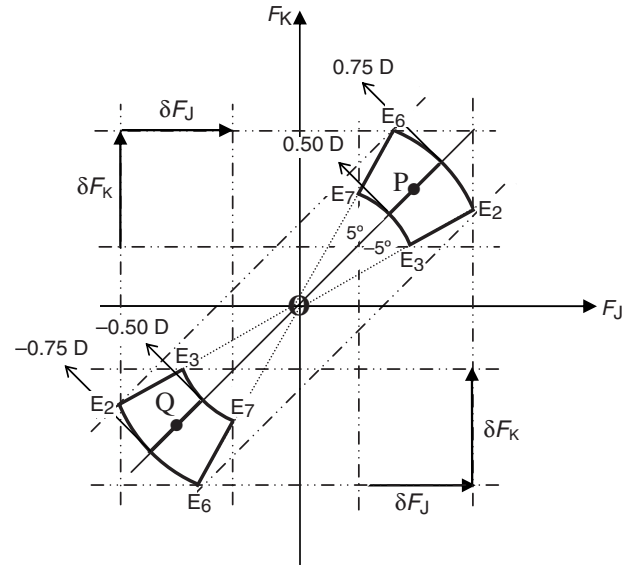


Fig. 4 These error cells for scalar powers in an antistigmatic plane are the image of cells in Fig. 2 that straddle lines at 22.5 and 112.5 deg, respectively. The two radial lines E₆E₇ and E₂E₃ and the arcs E₇E₃ and E₆E₂ of the same circle form the centrosymmetric error cells. The radii of the arcs define the bounds that the cross-cylinder lens powers assume. The lines define the extreme values that the direction of the cross-cylinder lens assumes about a particular axis position. Polar coordinates of points E₇ and E₂ enable one to find the rectangular coordinates $(F_J; F_K)$ of the cell points and the corners of the surrounding rectangles. The cells are two of many tangential neighbors in the quadrants of an antistigmatic plane, and the labeled points satisfy Eqs. (1). The difference between tangent, arc, and chord becomes less significant for greater magnitude cylinder powers and smaller errors in axis angle. The exact $F_J(F_K)$ values are in the interval between the abscissas of E₂ and E₃ (the ordinates of E₆ and E₇), and δF_J (δF_K) denote the lengths of the intervals. Rotation of the cell through 90 deg about an axis emanating from the paper at the origin allows what has been described for F_J and δF_J to be interchanged with what has been valid for F_K and δF_K and vice versa.

der lenses as compensation and a cross-cylinder lens as a probe, this method uses the cross-cylinder lens as both probe and compensation. The cross-cylinder options in Figs. 3(c) and 3(d), intermediate handle positions, and an interval of principal powers are contained in the geometry of Fig. 3(a) in an antistigmatic plane.

Similar cross-cylinder figures where handles rotate, lenses flip, and powers change emanate from Fig. 4. Flipping the lens takes one across the circle diameter; turning the lens takes one around the circle.

In the intervals $[-0.75, -0.50] D$ and $[0.50, 0.75] D$, we identify radii of concentric arcs on opposite sides of the origin. Examples are seen in Fig. 3(a). The radii of the arcs represent cross-cylinder powers with alternate signs. The powers are presented to the patient as the handle is flipped to counteract cross-cylinder powers of the ametropia. In particular, the points P and Q at the radii $-5/8 D$ and $5/8 D$ in Fig. 3(b) represent the potential power of the opposite sign presented to neutralize^{2,3} this cross-cylinder component of the ametropia.

In Fig. 3(a) the powers at P and Q and their location on the F_J axis are now known. From Fig. 3(a) the magnitude of F_K at E₂ and E₆ is $(0.75 \sin 5) D$. The "true" value of F_K

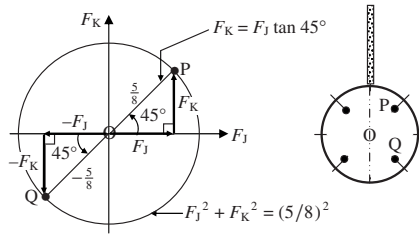


Fig. 5 This is a circle for which Eqs. (1) and (2) are the parametric equations. The parameters are the clinical quantities A and F_C . Elimination of one of the parameters yields the equation of a diameter $F_K = F_J \tan 2A$ of the circle $F_J^2 + F_K^2 = F_C^2/4$ in terms of the astigmatic powers F_J and F_K that have zero spherical equivalent or are Jackson cross-cylinders. Powers where dioptric power matrices have zero trace are to be found in this plane of symmetric power space. The cross-cylinder lens with its handle indicates how meridians with opposite powers alternate and coincide with radius OP and radius OQ .

about 0 deg is in the symmetric interval $(-0.75 \sin 5, 0.75 \sin 5)$ D. The length of this interval is $\delta F_K = 1.50 \sin 5$ D. The lower bound for F_J occurs where the arcs E_2E_6 intersect the axis, namely, -0.75 D. The upper bound is the F_J value at E_3 (or E_7). The asymmetric interval for the cell about $F_J = -0.625$ D at Q [see Eqs. (1)] is $(-0.75, -0.50 \cos 5)$ D. The length of this interval is $\delta F_J = (0.75, -0.50 \cos 5)$ D.

Sphere, cylinder, and axis cannot be optimized or suitably processed. However, in the example the astigmatic coordinates of an error cell are optimized under the estimated constraint.

The error cells in Fig. 3(a) contain all hypothetical powers that would be rounded to the transposed cylinder powers 1.25×0 and -1.25×90 (as well as -1.25×0 and 1.25×90) that keep the circle of least confusion on the retina.

Let points Q and P in Fig. 2 move to the right to vertical lines at 22.5 deg. The powers of cylinder 1.25×22.5 and -1.25×22.5 D are transformed to the rectangular astigmatic coordinates F_J and F_K ⁸ of the dioptric power matrix via

$$\begin{pmatrix} F_J \\ F_K \end{pmatrix} = -\frac{1.25}{2} \begin{pmatrix} \cos 2(22.5) \\ \sin 2(22.5) \end{pmatrix} = \frac{5}{8\sqrt{2}} \begin{pmatrix} -1 \\ -1 \end{pmatrix} D \text{ and}$$

$$\begin{pmatrix} F_J \\ F_K \end{pmatrix} = -\frac{-1.25}{2} \begin{pmatrix} \cos 2(22.5) \\ \sin 2(22.5) \end{pmatrix} = \frac{5}{8\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} D. \quad (2)$$

Equations (2) contain the powers on opposite sides of a diameter QP as similar right-angled triangles with a common vertex O in Fig. 5. Polar coordinates $(5/8; 45)$ that identify point P emanate from axis and cylinder $(22.5; -1.25)$.

The diameter QP in Fig. 3(a) has rotated through 45 deg for the same circle as in Eqs. (2), and is illustrated in Figs. 4 and 5. The polar coordinates of Q are then $(-5/8; 45) = (5/8; 225)$. A circle with diameter QP as in Fig. 5 has the same equation as an analogous circle in Fig. 3(a), $F_J^2 + F_K^2 = (5/8)^2$, but the diameter, determined by the axis, has the equation $F_K = F_J$.

The coordinates for each of the transposed cylinder powers have invariant astigmatic powers under transposition.

As the cross-cylinder handle along a line at 45 deg to diameter QP is flipped, P and Q at the radii $-5/8$ D and $5/8$ D represent the potential cross-cylinder power of the opposite sign presented to the patient^{2,3} for this component of the ametropia.

Let the error cells in Fig. 2 move to the right to straddle lines at 22.5 and 112.5 deg. Then the cells in Fig. 3(a) rotate to where cells straddle a line at 45 deg in Fig. 4. For each of these error cells, the extreme values of the rectangular coordinates F_J and F_K differ from those of Fig. 3(a). Thus the intervals and lengths need to be calculated anew. Error cells contain a set of points defined by a discrete power, where cross-cylinder lens meridians and their powers intersect. Let the handle of the cross-cylinder be along a line at 45 deg to PQ and the other diameters in Fig. 4. As the practitioner flips the handle powers of cross-cylinders from diametrically opposite error cells, signs and meridians are presented to the patient.

Assume that all of the many pure cylindrical powers and their axes in the error cells in Fig. 2 around the single powers at P and Q and their transposes are translated to symmetric dioptric power space. Then a family of diameters intersects its family of concentric circles so that concentric arcs and diameters define error cells. Diameters and concentric arcs or circles are patterns with cylindrical symmetry that are expected in such data. Any other patterns in a plane provide characteristics deemed worthy of further investigation. In an astigmatic plane, error cells are of the form and shape seen in Figs. 3(a) and 4. We have demonstrated that the orientation of the cells changes for different cylinder axes.

Error cells in Fig. 2 are at 0 and 90 deg, and in Fig. 3(a) their images straddle the F_J axis. Suppose the cells in Fig. 2 straddle lines about 45 and 135 deg, respectively. Then, in an astigmatic plane, an error cell straddles the F_K axis instead of the F_J axis. Although the discussion for F_J and δF_J may then be interchanged with the aforementioned for F_K and δF_K and vice versa, in general, cells need to be treated individually for angles, since a treatment of a single cell in the sample that covers all cases adequately does not exist. We have considered a representative sample of error cells at positions in an astigmatic plane.

4 Discussion and Conclusion

The shape of error cells has been tailored to include powers that are well defined with respect to the circle of least confusion and the patient's retina. Error cells are a set of probable compensatory powers about a discrete power that one may use to assess *just-noticeable differences* for refractive compensation and plan which powers to include in the research project. Error cells could contain *over-refractions* for patients with newly implanted *intraocular lenses*. The power of the over-refraction in air is at the center of the cell. Other possible over-refractions that keep the circle of least confusion near enough to the retina are located in the cell. Once one eye has been refracted and the order of the compensation is known, measurements for the other eye may be adjusted accordingly before the implant lens is ordered. Error cells give a fairly complete picture of potential compensations.

An interval of cross-cylinder lens powers and an interval of their axis positions have been modeled in an astigmatic

plane by coordinate geometry and properties of the dioptric power matrix. The point of departure was a set of possible lens compensations about discrete and corresponding powers of lenses in a trial frame. Intervals conveyed to data in symmetric power space influence their accuracy and the confidence one may place in them. Bounds were analyzed and optimized, and powers obtained by calculation were ranked. Optimization with the usual calculus routines produced no extreme values in multivariate power matrix coordinates. This work showed via geometry how bounds postcomputation allow intervals of individual antistigmatic coordinates of the power matrix to be obtained for thin cylinder lens powers and their axes.

For reported cylinder powers at specific axes, the cylinder axis and power were considered as the coordinates in Table 1 in the plane of Fig. 2. The plane was subdivided into a mesh of cells with a clinical measurement at the center. Rectangular cells with the same shape, size, form, and orientation were chosen and drawn for positive and negative cylinder powers. Coordinates of the centers and corners of such cells were determined. Every cylinder axis and power value on or in an error cell in an axis-cylinder plane was imaged onto cells in an antistigmatic plane. The lines that were perpendicular premapping became perpendicular lines and arcs postmapping. It is interesting to note the number of cells halved after mapping as a result of invariance of cells under spherocylindrical transposition.^{1,6} Numerical examples illustrate how to determine upper and lower bounds. Bounds and intervals of zero cylinder powers at all axis angles complement bounds and intervals about nonzero cylinder powers in an antistigmatic plane.

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