OE Optical Engineering

COMMENTARY

Commentary: aliasing, imaging, and money

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ABSTRACT. Aliasing is an unnecessarily confusing topic in imaging and in general. This commentary's motivation, in the first half, is to alleviate the confusion and remove the mystery of aliasing as well as provide a basic understanding and appreciation for sampling without the need for Fourier theorems or other higher-level mathematics. The second half discusses examples of where aliasing plays an important role in imaging.

 [DOI: 10.1117/1.OE.63.3.030401]

Keywords: aliasing; aliasing errors; sampling frequency; imaging; convolution; neural networks; money

Paper 20231277C received Dec. 30, 2023; revised Feb. 27, 2024; accepted Mar. 11, 2024; published Mar. 21, 2024.

It occurred to me after writing two articles[1,2] on aliasing, and its quantification for optical imaging systems in particular, that aliasing in general is somewhat of a mystery to many. I surmise this is because the mathematics is brought up early on and well before the issue is properly appreciated. This seems to turn people off, leading to a common response to the image sampling and aliasing question of, "it's just photons in a bucket," which is true but not relevant.

In the initial half of this commentary, I aim to dispel the confusion and explain aliasing generically using a commonplace but exaggerated example since discretizing or sampling is virtually everywhere and aliasing is its shadow. While I used optical imaging as the vehicle on how to quantify aliasing generally,[1,2] and whether you are enthused by optical imaging or sciences in general or neither, my objective is to explain what aliasing is and why it is so important in life without using mathematical equations. In the latter half of the commentary, examples are provided to show how aliasing error plays a role in the digital imaging chain with just a couple equations to provide finer details that are not referenceable.

To explain aliasing, let us choose the universally appreciated quantity that everyone is intimately familiar with, money spanned over time, in place of photons spanned spatially as in imaging. Consider entering a bank for a loan, an occurrence many are painfully aware of, and you approach a loan officer to fill out the necessary paperwork. After having filled in your full name and social security number repeatedly, you are asked for the last 12 months of your bank account value. Let us assume for the purposes of this exercise that there is only one true account value or balance for the day and there are 30 days in a month. The loan form is set up to enter (display) the balance only for the first day of every month. This happens to be very fortunate for you because while on the first day of every month you have a $100,000 balance, on the second day of every month, you make a withdrawal of $99,999, leaving only a balance of $1. Then on the last day of the month, you deposit $99,999 such that on the first day of the next month, you have again $100,000. This goes on month after month. What you did with the money from the second day to the last day of the month is not to be revealed, but nevertheless, you can write in $100,000 on each line of the form and feel comfortable signing at the bottom.

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The loan officer pulls together the paperwork and presents it to the bank manager for signoff. The bank manager is just about to do so noticing that the average balance on the form appears or interpolates[1] to be $100,000, but then asks the loan officer one simple question, "Could you tell me what the average balance is if you include the value on the 15th of every month?" As it stands, the form has room for only one displayed value on the first of the month, but the question is, what if we display or sample at twice that rate? The implication of the bank manager's question is the balance is not static (zero frequency, i.e., DC), and there are balance fluctuations (higher frequencies). If a sample is taken on the 15th of every month, we are attempting to capture what is routinely called the first fundamental frequency[3] Fourier component, which has a frequency of $\frac{1}{30}$ days^{-1}. When the 15th is not sampled, the presumption is the account has nominally $100,000 on the 15th (first fundamental frequency Fourier component folds over and masquerades as the zeroth frequency component), but we know this to be false or disinformation[1,2] because on the 15th there is only $1. If the loan officer included the 15th, the average balance would be more like $50,000. The true balance average is $100,029/30 or $3334, which is the same as sampling every day given the assumptions of our exaggerated example. While the sampled balances are exact, the problem comes to bear when averaging (a mathematical operation) is applied to the balances, followed by an assessment. The reason for the difference in the averages (true balance, loan form, bank manager's question) is there are higher frequencies in the balance that are not being considered in the loan form or the bank manager's question leading to different and incorrect assessments. This incorrect assessment, coined as aliasing, is quantified in my recent articles using all the aliased frequency components, not simply the first fundamental (Fig. 1).

Essentially, the aliasing error is a measure of how well neighboring samples are correlated. If not well correlated as in our exaggerated example, then many mathematical operations applied to the sampled data, such as an average or interpolation[4,5] or Fourier transform[1] or convolution (e.g., neural networks), yield poor or incorrect assessment(s). In imaging, a moiré pattern is a result of aliasing in its most pronounced form, as shown figure 6.26 of Ref. 6 and figure 15 of Ref. 7. The pattern is generated by successive low or high samples being captured due to a high-frequency, somewhat periodic object giving the appearance that there are low and high regions (low frequency) in the image as our eye is performing an interpolation between samples (similar to the bank loan example above). Further, a de-convolution or Wiener filter application, often used to sharpen an image by boosting high(er) frequencies, does not escape aliasing errors.[8] If aliasing is not considered in the design of the filter, then boosting the aliased frequencies, such as the frequency components producing a moiré pattern, can cause even more errors or disinformation in the image.[1]

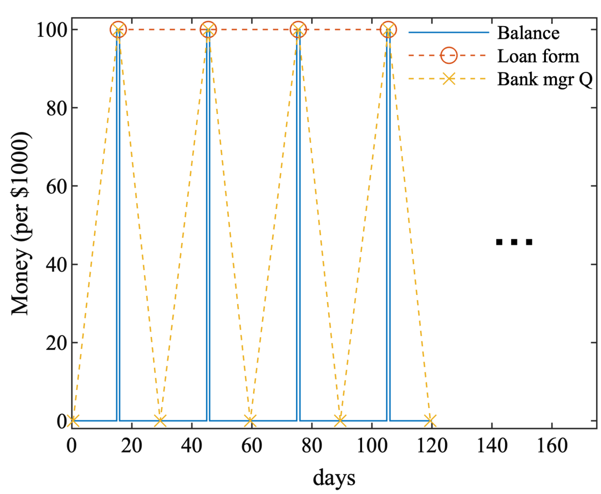

Fig. 1 Balance, loan form entries, and bank manager question data. The dashed lines between the samples on the latter two are a form of interpolation sometimes performed only by eye, as with the bank manager when viewing the loan form. The loan form data appear as a bias of $100,000, whereas the bank manager's question data are a bias of $50,000 plus a wave with amplitude $50,000 and frequency $\frac{1}{30}$ days^{-1}, which is the first fundamental frequency.

Convolution neural networks, which basically convolve the item being searched with the sampled image looking for a match,[9] are not immune to aliasing errors.[10] This sampled image error impacts the convolution output. Simplistically, the search consists of the convolution output reaching a particular threshold at a particular spatial position giving a probability of detection or false alarm. The convolution has errors due to aliasing[1] from the sampled image and can be seen in the frequency domain by utilizing the convolution theorem.[3] Mathematically in the Fourier domain, the searched item, $s(x_s)$, in the sampled image is multiplied by the sampled image, $i(x_s)$, or

$$\mathcal{F}_D\{s(x_s) * i(x_s)\} = S(\sigma_s)\tilde{I}(\sigma_s), \tag{1}$$

where x_s and σ_s are the one-dimensional sampled position and frequency, respectively, $\mathcal{F}_D\{\cdot\}$ is the discrete Fourier transform, and the sampled image spectrum $\tilde{I}(\sigma_s)$ has aliasing errors. Exposing the aliasing error embedded in the convolution output gives

$$S(\sigma_s)\tilde{I}(\sigma_s) = S(\sigma_s)(I(\sigma_s) - I_\epsilon(\sigma_s)) = S(\sigma_s)I(\sigma_s) - S(\sigma_s)I_\epsilon(\sigma_s), \tag{2}$$

where $I(\sigma_s)$ has no aliasing error and $I_\epsilon(\sigma_s)$ as a function of sampled frequency represents the error in the sampled image due to aliasing. Reference 1 calculates the aliasing error (ϵ) as a summation over frequency and suggests the summation could be broken into frequency bands if desired. This aliasing error, linked to the last term on the right $S(\sigma_s)I_\epsilon(\sigma_s)$, represents a fundamental limit error in a convolution neural network.

On a grander scale, aliasing error[1] plays a role in information theory. This error represents a relationship between the information available to be sampled and the amount of disinformation[1] after sampling. The aliasing error is then fundamentally linked to Shannon's information theory.[11,12]

How did the loan officer respond to their manager's request? you ask. The way any good officer would: "There is no place on the form to put the account balance for the 15th."

Code and Data Availability

There is no code or data publicly available.

Acknowledgments

Parts of this commentary were supported by the optics consulting company Golden Gate Light Optimization, LLC. Additional thanks to Richard L. Kendrick of Laurel Creek Optics, LLC who provided this fitting quote from Edward R. Morrow, "To be persuasive, We must be believable, To be believable, We must be credible, To be credible, We must be truthful." Thanks to my wife, Georgina, and to my parents, Mavis and John.

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