Parrondo Games with Strategy

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ABSTRACT

The notion of a strategy in the multi-player version of the Parrondo game is reviewed. We calculate the gain for the greedy strategy as a function of the number N of players, including exact analytic results for $N < 4$ and in the limit $N \to \infty$. We show that the greedy strategy is optimal for $N = 1$ and $N = 2$ but not for $N = 3$. In the limit $N \to \infty$ our analysis reveals a very rich behavior including the possibility of phase transitions as a function of the chosen strategy.

Keywords: Parrondo paradox, strategy, game theory, Brownian motor

1. INTRODUCTION

Random or periodic switching between fair games may no longer be fair. This surprising observation, known under the name of Parrondo paradox, derives from a deeper and less surprising statement in statistical mechanics: a system that undergoes random or periodic switching between two equilibrium dynamics is no longer at equilibrium. More specifically, equilibrium dynamics is characterized by detailed balance, implicating that any transition between two states of the system and the reverse transition are equally probable. This very strong probabilistic symmetry implies in particular the absence of fluxes -the analogue of fairness in the games- but is in no ways guaranteed by it. Switching between two dynamics with detailed balance can and typically will break detailed balance and produce fluxes. This phenomenon has been particularly well studied recently in the context of Brownian motors, with the flashing ratchet as one of the prototypes for rectification of thermal fluctuations.[1]
Another type of equilibrium was introduced by von Neumann and Morgenstern in their groundbreaking formulation of game theory.[2] Players are allowed to choose a probabilistic (so-called mixed) strategy in relation to a pre-specified pay-off table between competing partners. The equilibrium strategy is optimal in the sense that no player can change his strategy without risking to be worse of. More recently, more complicated dynamical and iterated versions of this set-up have been investigated. We cite the open entry competition for algorithms fighting on the basis of the iterated prisoner's dilemma[3] (with the surprising feature of the simple tit-for-tat strategy comes out as a winner), and the theoretical activity with respect to the minority game, in which participants vote between two alternatives with the aim of belonging to the minority.[4] Finally we mention the huge engineering field of plant control, with as main issue the optimization of the output usually in the presence of conflicting requirements or constraints (cfr. the related issue of finding free energy minima in frustrated systems).
Recently a variation of the Parrondo paradox was introduced that provides a toy model in which all these issues appear. The Parrondo game is now played by a group of players with the collective aim of maximizing their total gain. The players have, for each new game, the freedom to choose -after mutual consultation- which one is played next. They can make this choice using a mixed strategy (i.e. according to some probability distribution). In engineering parlour the various games can be compared to different operational states of a plant. Fairness in this context implies that the factory does not function properly -the output is zero- when only one mode of operation is followed. The question of identifying the optimal strategy turns out to be surprisingly difficult, even when we restrict ourselves, as we will do here, to the case of Markovian games. The actual state of the players thus completely determines the probabilities for the events in the next game. At first, one might think that in this case the "greedy" strategy, which chooses at each turn the game that will generate the maximum total gain, will be optimal. It turns out that this is not the case.[5] The reason is that the choice of the game will also influence the state of the players, which may compromize the potential gain in the following game. To illustrate and clarify this issue, we present an exhaustive calculation of the gain for all possible strategies. This can be done either in the limit of a small number of players -we present results for up to 4 players- or in the limit of an infinit number of players using a mean field approach. In the latter limit, our analysis reveals a very rich behavior including the possibility of bifurcations, including bistability and abrupt transitions, in the gain of the players.

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2. STRATEGY IN THE PARRONDO GAME WITH A SINGLE PLAYER

For simplicity we will use the games from the orginal Parrondo paradox.[6,7] In game A, the player has a probability p_1 to win and probability $1 - p_1$ to lose. In game B, the probabilities depend on the capital $X(t)$ of the player: when $X(t)$ is not a multiple of 3, the probability to win is p_2, versus $1 - p_2$ for the probability to lose. On the other hand, for $X(t)$ a multiple of 3, the probability to win is p_3, and the probability to lose is $1 - p_3$. Both games taken separately are fair for the values $p_1 = 1/2, p_2 = 3/4$ and $p_3 = 1/10$.

Before turning to the discussion of strategies involving N players, we first illustrate the idea and results for the case of a single player. In the original Parrondo game, the type of game (A or B) is chosen at random or periodically. The main difference for the Parrondo game with strategy is that the player is allowed to choose which game he wants to play. After the game is played, the capital $X(t)$ is updated according to the outcome, and one moves to the next game. The main issue for the player is to come up with a good strategy: when selecting a game, he should use optimally all the relevant information, which for a single player (and Markovian games) is his actual capital modulo 3, a value which we will refer to as the (internal or reduced) state. In anticipation of the multi-player case, we will represent this state by a vector, namely $[1,0,0]$, $[0,1,0]$ or $[0,0,1]$, depending on whether the capital modulo 3 is equal to 0, 1 and 2 respectively. The most general (mixed) strategy is then defined by the probability distribution $s_{[1,0,0]}$, $s_{[0,1,0]}$ and $s_{[0,0,1]}$ to choose game A, when being in the respective states $[1,0,0]$, $[0,1,0]$ and $[0,0,1]$. Game B is then selected with probability $1 - s$. The efficiency of a strategy will reflect itself in the long-time average gain per game, denoted as G_1. The sub-index refers to the number of players which is one in the present case. To calculate the latter quantity, we need to study the statistics of the state of the player as it is induced by his strategy.[6] The probabilities $P_{[1,0,0]}(n), P_{[0,1,0]}(n)$ and $P_{[0,0,1]}(n)$ to observe the state $[1,0,0]$, $[0,1,0]$ or $[0,0,1]$ after playing the n-th game, obey the following Master equation (using an obvious vector notation):

$$\mathbf{P}(n+1) = \mathbf{R}.\mathbf{P}(n)\,. \tag{1}$$

The elements of $\mathbf{R}$ are the transition probabilities between the states, see Table 1 for details. When playing game A and B separately $\mathbf{R}$ reduces to either $\mathbf{R}_A$ or $\mathbf{R}_B$ respectively, given by:

$$\mathbf{R}_A = \begin{pmatrix} 0 & 1-p_1 & p_1 \\ p_1 & 0 & 1-p_1 \\ 1-p_1 & p_1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{R}_B = \begin{pmatrix} 0 & 1-p_2 & p_2 \\ p_3 & 0 & 1-p_2 \\ 1-p_3 & p_2 & 0 \end{pmatrix}\,. \tag{2}$$

The transition matrix describing the (mixed) strategy game is then given by:

$$\mathbf{R} = \mathbf{R}_A.\mathbf{S} + \mathbf{R}_B.(\mathbf{1} - \mathbf{S})\,, \tag{3}$$

where $\mathbf{S}$ is the (diagonal) strategy matrix:

$$\mathbf{S} = \begin{pmatrix} s_{[1,0,0]} & 0 & 0 \\ 0 & s_{[0,1,0]} & 0 \\ 0 & 0 & s_{[0,0,1]} \end{pmatrix}\,. \tag{4}$$

The average gain upon playing the n-th game is given by:

$$\begin{aligned} G_1(n) &= P_{[1,0,0]}(n)\left[s_{[1,0,0]}(2p_1 - 1) + (1 - s_{[1,0,0]})(2p_3 - 1)\right] \\ &+ P_{[0,1,0]}(n)\left[s_{[0,1,0]}(2p_1 - 1) + (1 - s_{[0,1,0]})(2p_2 - 1)\right] \\ &+ P_{[0,0,1]}(n)\left[s_{[0,0,1]}(2p_1 - 1) + (1 - s_{[0,0,1]})(2p_2 - 1)\right]\,. \end{aligned} \tag{5}$$

In the following we will focus entirely on the long-time steady state results. Hence, we only require the steady state solution $\lim_{n\to\infty} \mathbf{P}(n) = \mathbf{P}^{st}$ being the normalized eigenvector with eigenvalue 1 of the transition matrix $\mathbf{R}$:

$$\mathbf{R}.\mathbf{P}^{st} = \mathbf{P}^{st}\,. \tag{6}$$

Table 1. List of all possible transitions between the configurations for one player, the corresponding change in capital and the transition probability.

Transition	Capital	Transition probability game A	game B
$[1,0,0] \to [0,1,0]$	$+1$	p_1	p_3
$[0,0,1]$	-1	$1-p_1$	$1-p_3$
$[0,1,0] \to [0,0,1]$	$+1$	p_1	p_2
$[1,0,0]$	-1	$1-p_1$	$1-p_2$
$[0,0,1] \to [1,0,0]$	$+1$	p_1	p_2
$[0,1,0]$	-1	$1-p_1$	$1-p_2$

Lengthy calculations - that are most easily performed using symbolic manipulators - lead to the following result, where we used the abbreviations: $s_0 = s_{[1,0,0]}$, $s_1 = s_{[0,1,0]}$ and $s_2 = s_{[0,0,1]}$:

$$\begin{aligned} G_1 \;=\; & 3\left[(1+p_3(s_0-1)-p_1 s_0)\,(p_1 s_1-1) - {p_2}^2(1+2p_3(s_0-1)-2p_1 s_0)\,(s_1\right. \\ & -1)(s_2-1) + p_1\,(1+p_3(s_0-1)+2p_1\,(p_3+p_1 s_0 - p_3 s_0)\,s_1 - p_1(s_0 \\ & +s_1))\,s_2 + p_2\,(2-s_1-s_2+p_3(s_0-1)\,(2-s_2-2p_1 s_2+s_1(4p_1 s_2-1 \\ & \left.-2p_1)) + p_1\,(2s_1 s_2 - s_1 - s_2 + s_0\,(s_1+2p_1 s_1 - 2 + s_2 + 2p_1 s_2 - 4p_1 s_1 s_2)))\right] / \\ & \left[3 - p_1\,(s_0+s_1) + {p_2}^2(s_1-1)(s_2-1) + p_1\,(p_1 s_0 s_1 - s_2 + p_1(s_0+s_1)s_2)\right. \\ & -p_3(s_0-1)(p_1 s_1 + p_1 s_2 - 1) + p_2\,(s_1+s_2-2+p_3(s_0-1)(s_1+s_2-2) \\ & \left.+p_1\,(s_1+s_2-s_0(s_1+s_2-2)-2s_1 s_2))\right]. \end{aligned} \tag{7}$$

Replacing $p_1 = 1/2, p_2 = 3/4$ and $p_3 = 1/10$, the final result reads:

$$G_1 = \frac{6\,[10s_0 - 3(s_1+s_2) - 2s_1 s_2 - 2s_0(s_1+s_2) + 2s_0 s_1 s_2]}{169 + 16s_0 + 3s_1 + 3s_2 + 5s_1 s_2 - 8s_0(s_1+s_2)}. \tag{8}$$

The optimal strategy, giving the highest value of G_1, is found to be a pure strategy, namely $s_{[1,0,0]} = 1$ and $s_{[0,1,0]} = s_{[0,0,1]} = 0$. The corresponding average gain per game is $G_1 = 60/185 \approx 0.3243$. Not surprisingly, this strategy has a very simple interpretation: whenever the player is in configuration $[1,0,0]$, i.e. when B is a losing game, the player chooses game A (with probability 1). Otherwise, that is in the states $[0,1,0]$ or $[0,0,1]$, the winning game B is chosen. Hence, the optimal strategy is in this case identical to the greedy strategy discussed in the introduction. For comparison, note that the original Parrondo game corresponds to the strategy $\{s_{[1,0,0]}, s_{[0,1,0]}, s_{[0,0,1]}\} = \{1/2, 1/2, 1/2\}$, in which the player chooses at random between the two games. For this strategy the average gain is $G_1 = 18/709 \approx 0.0254$, roughly a factor 13 smaller than for the optimal strategy!

3. PARRONDO GAME WITH STRATEGY FOR N PLAYERS

The generalization to N players is now straightforward. At each timestep, a game is choosen and is then played by all players. The strategy that is used to select a game will now depend on the capitals modulo 3 of all players. The collective state can now be represented by $[N_0, N_1, N_2]$ where N_i is the number of players who's capital modulo 3 is equal to $i = 0, 1, 2$ respectively ($N_0 + N_1 + N_2 = N$). The probability for the players to be in configuration $[N_0, N_1, N_2]$ after the n-th game is played, will be denoted by $P_{[N_0,N_1,N_2]}(n)$. A strategy is defined by the probabilities $s_{[N_0,N_1,N_2]}$ to choose game A when being in the corresponding state. The (steady state) analogue of Eq.(5) for N players is then:

$$G_N = \frac{1}{N} \sum_{N_0,N_1,N_2} P^{st}_{[N_0,N_1,N_2]} \left[s_{[N_0,N_1,N_2]}[2Np_1 - N] + (1 - s_{[N_0,N_1,N_2]})[2N_0 p_3 + 2(N_1+N_2)p_2 - N] \right], \tag{9}$$

where G_N is the average gain per player and per game. Note that the summation over N_0,N_1 and N_2 runs over all $(N+1)(N+2)/2$ different configurations. The stationary distribution $P^{st}_{[N_0,N_1,N_2]}$ is found as the normalized eigenvector of eigenvalue one of the transition matrix $\mathbf{R}$, which is now an $(N+1)(N+2)/2 \times (N+1)(N+2)/2$ matrix.
To illustrate the procedure, consider the $N = 2$ player game. In this case, there are a total of 6 different configurations, namely:

$$\begin{array}{ccccc} [2,0,0] & , & [1,1,0] & , & [1,0,1] \ , \\ [0,2,0] & , & [0,1,1] & , & [0,0,2] \ . \end{array} \tag{10}$$

Table 2 summarizes all possible transitions between the configurations, the corresponding change in capital, and the transition probability.
The matrix $\mathbf{R}_B$ is thus:

$$\mathbf{R}_B = \begin{pmatrix} 0 & 0 & 0 & (1-p_2)^2 & p_2(1-p_2) & p_2p_2 \\ 0 & p_3(1-p_2) & p_2p_3 & 0 & (1-p_2)^2 & 2p_2(1-p_2) \\ 0 & (1-p_2)(1-p_3) & p_2(1-p_3) & 2p_2(1-p_2) & p_2p_2 & 0 \\ p_3p_3 & p_2p_3 & p_3(1-p_2) & 0 & 0 & (1-p_2)^2 \\ 2p_3(1-p_3) & 0 & (1-p_2)(1-p_3) & 0 & p_2(1-p_2) & 0 \\ (1-p_3)^2 & p_2(1-p_3) & 0 & p_2p_2 & 0 & p_2p_2 \end{pmatrix} . \tag{11}$$

$\mathbf{R}_A$ is found by replacing p_2 and p_3 in $\mathbf{R}_B$ with p_1. The calculation of the stationary distribution $\mathbf{P}^{st} = (P^{st}_{[2,0,0]}, P^{st}_{[1,1,0]}, P^{st}_{[1,0,1]}, P^{st}_{[0,2,0]}, P^{st}_{[0,1,1]}, P^{st}_{[0,0,2]})^T$ can be handled by a symbolic manipulator. The final result for G_2 is rather lengthy and not reproduced here. An exhaustive search over all pure strategies allows to identify the following optimal strategy:

$$\{s_{[2,0,0]}, s_{[1,1,0]}, s_{[1,0,1]}, s_{[0,2,0]}, s_{[0,1,1]}, s_{[0,0,2]}\} = \quad \{1,1,1,0,0,0\}, \tag{12}$$

and the corresponding value for $G_2 = 1312/5913 \approx 0,2219$. Furthermore this strategy is again identical to the greedy strategy: whenever the average gain for playing game B is negative, game A is choosen. This is the case for the configurations $[2,0,0], [1,1,0]$ and $[1,0,1]$, with an expected gain in game B equal to $-8/5$,$-3/10$ and $-3/10$ respectively. Note also that the average optimal gain is only about $2/3$ of that of a single player. This is obviously due to the fact that the same game has to be chosen for all players leading to a conflict of interest and reduced pay-off. Note finally that for the original Parrondo game, $s = 1/2$, collective effects are immaterial and one recovers the well know average gain $G_1 = 18/709$.

So far we found that the greedy strategy is optimal for $N = 1$ and $N = 2$. This is, as we proceed to show next, no longer the case for $N \geq 3$. We first focus on the case $N = 3$. It is straighforward to repeat the above calculations (involving now 10×10 matrices) but the procedure and final expressions are very lengthy so we just review the salient results. The different configurations are now:

$$\begin{array}{ccccccccc} [3,0,0] & , & [2,1,0] & , & [2,0,1] & , & [1,2,0] & , & [1,1,1] \ , \\ [1,0,2] & , & [0,3,0] & , & [0,2,1] & , & [0,1,2] & , & [0,0,3] \ . \end{array} \tag{13}$$

Referring to the ordering of the states in (13), the greedy strategy corresponds to $\{1,1,1,0,0,0,0,0,0,0\}$ for which we find an average steady state gain per game and per player equal to:

$$G_3 = \frac{2317431670848}{4664732583395} \approx 0.1656\,. \tag{14}$$

However, the strategy $\{1,1,1,0,0,1,0,0,0,0\}$, which differs from the greedy strategy in choosing the neutral game A rather then the winning game B in configuration $[1,0,2]$ has a larger gain, namely:

$$G_3 = 45185912531/86483373591 \approx 0.1742\,. \tag{15}$$

In fact, the greedy strategy is only the third best one. The second best strategy is $\{1,1,1,0,1,1,0,0,0,0\}$, with average gain $G_3 = 5984181363/11678660585 \approx 0.1708$, differs in two configurations from the greedy one.

Table 2. List of all possible transitions between the configurations for two players, the corresponding change in capital and the transition probability.

Transition	Capital	Transition probability game A	game B
$[2,0,0] \to [0,2,0]$	$+2$	$p_1 p_1$	$p_3 p_3$
$[0,0,2]$	-2	$(1-p_1)^2$	$(1-p_3)^2$
$[0,1,1]$	0	$2p_1(1-p_1)$	$2p_3(1-p_3)$
$[1,1,0] \to [0,1,1]$	$+2$	$p_1 p_1$	$p_2 p_3$
$[1,0,1]$	-2	$(1-p_1)^2$	$(1-p_2)(1-p_3)$
$[1,1,0]$	0	$p_1(1-p_1)$	$p_3(1-p_2)$
$[0,0,2]$	0	$p_1(1-p_1)$	$p_2(1-p_3)$
$[1,0,1] \to [1,1,0]$	$+2$	$p_1 p_1$	$p_2 p_3$
$[0,1,1]$	-2	$(1-p_1)^2$	$(1-p_2)(1-p_3)$
$[0,2,0]$	0	$p_1(1-p_1)$	$p_3(1-p_2)$
$[1,0,1]$	0	$p_1(1-p_1)$	$p_2(1-p_3)$
$[0,2,0] \to [0,0,2]$	$+2$	$p_1 p_1$	$p_2 p_2$
$[2,0,0]$	-2	$(1-p_1)^2$	$(1-p_2)^2$
$[1,0,1]$	0	$2p_1(1-p_1)$	$2p_2(1-p_2)$
$[0,1,1] \to [1,0,1]$	$+2$	$p_1 p_1$	$p_2 p_2$
$[1,1,0]$	-2	$(1-p_1)^2$	$(1-p_2)^2$
$[2,0,0]$	0	$p_1(1-p_1)$	$p_2(1-p_2)$
$[0,1,1]$	0	$p_1(1-p_1)$	$p_2(1-p_2)$
$[0,0,2] \to [2,0,0]$	$+2$	$p_1 p_1$	$p_2 p_2$
$[0,2,0]$	-2	$(1-p_1)^2$	$(1-p_2)^2$
$[1,1,0]$	0	$2p_1(1-p_1)$	$2p_2(1-p_2)$

The fourth best strategy (average gain $G_3 = 284867630136/578009589745 \approx 0.1643$) is $\{1,1,1,0,1,0,0,0,0,0\}$. As a usefull check of our calculations, we also note that we find three different strategies that give an average gain of 0, namely $\{0,0,0,0,0,0,0,0,0,0\}$, $\{0,0,0,0,1,0,0,0,0,0\}$ and $\{1,1,1,1,1,1,1,1,1,1\}$. The first and the last strategy correspond to a choice of uniquely playing game B and game A respectively, and the zero gain reflects that these games are fair. The second strategy corresponds to playing game B all the time except in configuration $[1,1,1]$, where game A is chosen. The explanation why this does not affect the average gain is that game A will, when the players are evenly spread over the three states, maintain -statistically speaking- the uniform distribution over these states.

The analytic results for $N \geq 4$ are unwieldy, so we have resorted to numerical simulations to further investigate the performance of the greedy algorithm as N becomes larger, cf. Fig. 1. As expected G_N decreases further with increasing N and appears to converge for $N \to \infty$ to an asymptotic value larger than that of the random game. This suspicion will be confirmed by the analytic evaluation of $\lim_{N\to\infty} G_N$ for the greedy strategy, presented in the next section.

4. MEAN FIELD PARRONDO STRATEGY

The above calculations become more and more involved as N becomes larger, but a significant simplification takes place in the limit $N \to \infty$. In this limit we introduce the fractions $x_0 = N_0/N$, $x_1 = N_1/N$, and $x_2 = N_2/N$, of players that have a capital equal to 0, 1, and 2, modulo 3 respectively. Note that $x_0 + x_1 + x_2 = 1$. Once the next game to be played has been selected, the law of large number stipulates that these quantities obey a determistic equation of evolution identical to the equation for the probability of a single player. Hence for game A one has :

$$\mathbf{x}(n+1) = \mathbf{R}_A.\mathbf{x}(n)\,, \tag{16}$$

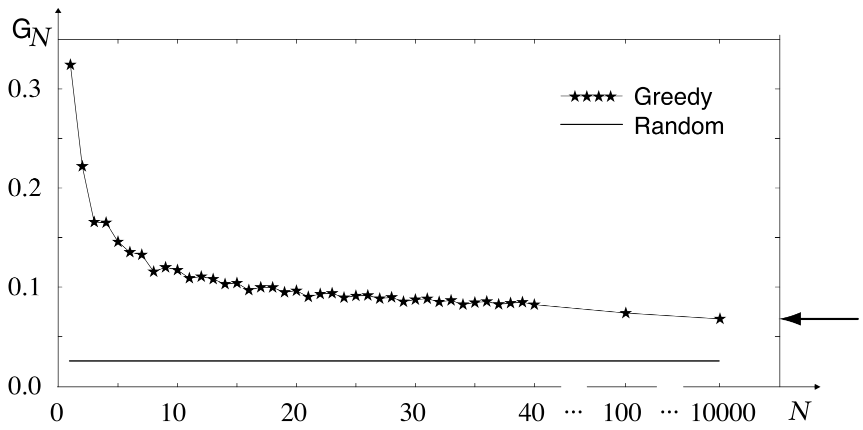


Figure 1. Comparison of the greedy strategy ($\star$) with the random strategy ($\diamond$) for a different number of players. The data for $N \leq 3$ is obtained from analytical calculations, those for $N \geq 4$ are obtained from simulations. The arrow points to the mean field result $G_\infty = 2416/35601 \approx 0.06786$

with $\mathbf{R}_A$ given by (2). This mapping has a unique and stable fixed point $x_0^A = x_1^A = x_2^A = 1/3$. The dynamics when game B is chosen is as follows

$$\mathbf{x}(n+1) = \mathbf{R}_B.\mathbf{x}(n)\,, \tag{17}$$

with $\mathbf{R}_B$ given by (2). The unique and stable fixed point of this map is : $x_0^B = 5/13$, $x_1^B = 2/13$, $x_2^B = 8/13$. The separate dynamics are thus linear maps converging exponentially fast to their respective unique fixed point. Non-trivial results arise when we introduce a strategy. In full analogy to the previous discussion, such a strategy is defined by the state dependent probability $s_{[x_0,x_1]}$ to select game A (when being in state $[x_0, x_1, x_2 = 1-x_0-x_1]$). $1 - s_{[x_0,x_1]}$ is the probability for game B. The dynamics for the strategy game is then as follows:

$$\mathbf{x}(n+1) = \left[\sigma(n)\mathbf{R}_A + (1-\sigma(n))\mathbf{R}_B\right].\mathbf{x}(n)\,. \tag{18}$$

where $\sigma(n)$ is a random variable equal to 1 with probability $s_{[x_0,x_1]}$ and 0 otherwise. We conclude that the resulting dynamics is in general a random and nonlinear map. Two limiting cases are worth mentioning. In the special case that $s[x_0, x_1]$ is a constant independent of $[x_0, x_1]$, eq. (18) represents a random linear map, a case which has received considerable attention in the literature on fractals since the invariant distibution is typically fractal or multi-fractal.[8] As an illustration we reproduce in Fig. 2 the numerically obtained support of the two-dimensional steady state probability in the case $s_{[x_0,x_1]} = 1/2$.

To make further progress, we turn to the other case which is of more interest to us here, namely we restrict ourselves to the case of pure strategies. A strategy is now defined by a boundary in the (x_0, x_1)-plane, separating the region where game A is played, $s_{[x_0,x_1]} = 1$, from the region where B is played, $s_{[x_0,x_1]} = 0$. The mapping (18) is then no longer random but becomes piece-wise linear. To fix the ideas and for comparison with previous results, we first focus on the greedy strategy in which game A is played when the expected gain in B is zero or negative, i.e., when $x_0(2p_3-1) + x_1(2p_2-1) + x_2(2p_2-1) \leq 0$. The boundary separating game A from game B is thus a straight line $x_0 = (1/2 - p_2)/(p_3 - p_2)$ or $x_0 = 5/13$ for the fair game choice $p_2 = 3/4$ and $p_3 = 1/10$. Game A is selected for $x_0 \geq 5/13$. To find the invariant state that is reached in the long time dynamics, we make the following observations. First, both pieces of the map are contracting. Hence the long-time dynamics has to take place on a subset of measure zero, possibly a fixed point, a periodic orbit or a chaotic trajectory. The second observation is that the fixed point of the A dynamics, $x_0^A = 1/3$, lies outside the region in which game A is played, while the fixed point of the B dynamics, $x_0^B = 5/13$, lies exactly on the boundary. We conclude that neither of them can be a stable fixed point of the greedy dynamics. The third observation is that any point belonging to the A region, $x_0 > 5/13$, is, upon playing game A, mapped onto the B region, implying that game A is directly followed by game B. A fourth similar observation can be made if we divide the B region, $x_0 < 5/13$, into subregions B' and B'', indicated by the dashed line in Fig. 3: B' is mapped onto B'', and B''

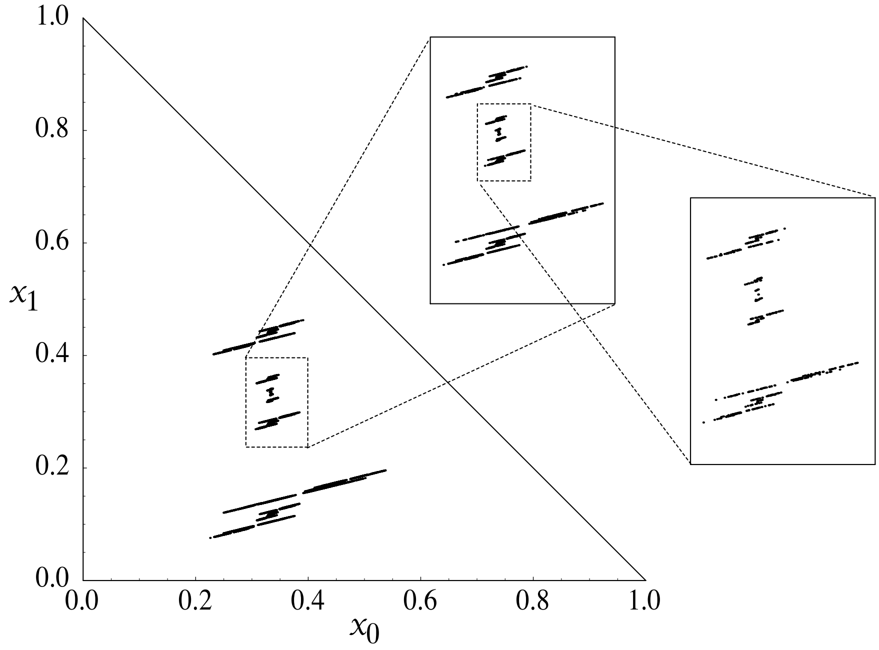


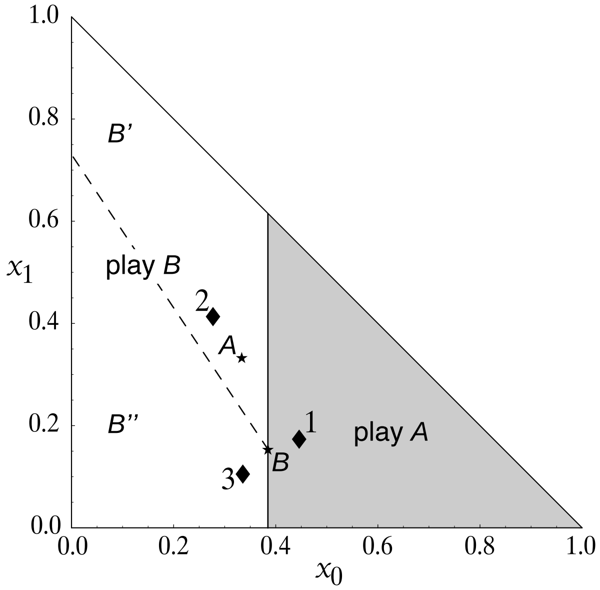
Figure 2. Two dimensional fractal support of the steady state probability distribution in the case $s_{[x_0,x_1]} = 1/2$. The blown up insets illustrate the self-similarity of the fractal.

onto the A region. We conclude that the greedy strategy leads to a game sequence built out of AB and/or ABB subsequences. One can now proceed to study the existence and stability of the fixed points corresponding to limit cycles of increasing complexity, e.g., $\ldots ABABAB \ldots$, $\ldots ABBABBABB \ldots$, etc.. For example, the limit cycle $\ldots ABABAB \ldots$ would correspond to an alternation between the fixed points of the matrix $\mathbf{R}_A.\mathbf{R}_B$ and $\mathbf{R}_B.\mathbf{R}_A$. These fixed points are unique and stable, namely $(3/13, 1/13)$ going over by the map A into $(5/13, 6/13)$, and back by the map B. Both points lie however in the region where the "wrong" game is played, and therefore this cycle is not compatible with the greedy algorithm. The only stable limit cycle compatible with the greedy dynamics that we could identify is the following one of period three:

$$\begin{array}{c} \{15871/35601, 6173/35601\} \\ \downarrow A \\ \{9865/35601, 14714/35601\} \\ \downarrow B \\ \{11945/35601, 3742/35601\} \\ \downarrow B \\ \ldots \end{array} \tag{19}$$

This cycle corresponds to a game sequence $\ldots ABBABBABB \ldots$. The corresponding average gains at each of these steps are $0 \to 4976/35601 \to 2272/35601$ with overall average $G_\infty = 2416/35601 \approx 0.06786$, in perfect agreement with the numerical results, cf. Fig. 1.

We turn to a last question of interest, namely whether there exists a strategy that beats the performance of the greedy strategy when $N \to \infty$. Building on our experience with the previous analysis, where the asymptotic dynamics are characterized by a limit cycle, we investigate this matter by proceeding in the reverse manner: we first identify a limit cycle with an average gain larger than the greedy strategy, and proceed to construct a strategy by choosing the boundary with the various points of the limit cycle sitting in the appropriate region of the (x_0, x_1) plane. The first problem was already solved in Ref. 9, where periodic sequences (such as for example $\ldots ABBABBABBABB \ldots$) of the single player game were studied. After an exhaustive search, for sequences with a period length up to 12, the period-5 sequence $ABABB$ comes out as the best. The five points of this



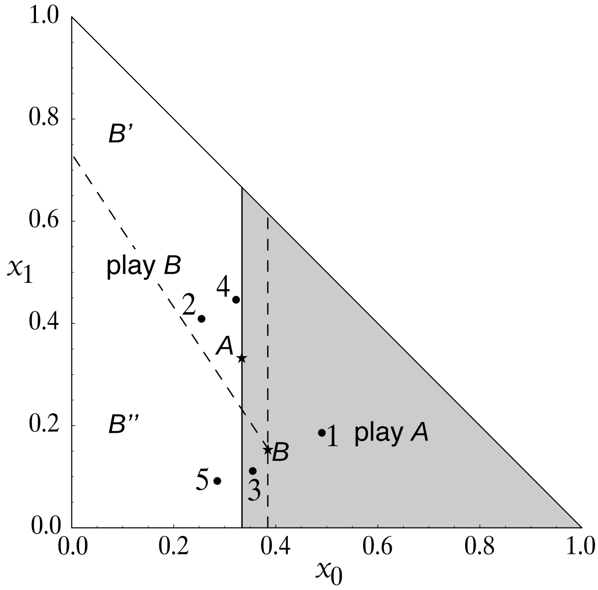


Figure 3. Sketch of the (x_0, x_1)-plane. Game A is played in the gray region, game B in the white region. The fixed points of the A and B dynamics are shown by a $\star$. Left panel: the full vertical line at $x_0 = 5/13$ shows the boundary line for the greedy strategy. The period-3 limit cycle is indicated by the ♦ symbol (the index refers to the order in which they follow each other). Right panel: the full vertical line at $x_0 = 0.334$ shows the boundary for the better strategy, while the dashed vertical line at $x_0 = 5/13$ shows the boundary line for the greedy strategy. The period-5 limit cycle is indicated by the • symbol (the index refers to the order in which they follow each other).

limit cycle are:

$$\begin{array}{c} \{4684919/9549529, 1762493/9549529\} \\ \downarrow A \\ \{2432305/9549529, 3893518/9549529\} \\ \downarrow B \\ \{3391159/9549529, 1049157/9549529\} \\ \downarrow A \\ \{3079185/9549529, 4250186/9549529\} \\ \downarrow B \\ \{2727665/9549529, 862958/9549529\} \\ \downarrow B \\ \cdots \end{array} \tag{20}$$

with corresponding average gains at each of these steps:

$$0 \to 1612768/9549529 \to 0 \to 771824/9549529 \to 2272/9549529\,,$$

and the overall average gain given by $G_\infty = 3613392/47747645 \approx 0.07568$. By plotting the points of the limit cycle in the (x_0, x_1) plane, it becomes immediately apparent that this period-5 limit cycle is realized when the boundary line of the greedy strategy is shifted to the left, i.e., to a value $x_0 \in]1/3, 11945/35601[$. The lower boundary $x_0 > 1/3$ is needed to ensure that the fixed point of A is still in the region where B is played, while the condition $x_0 < 11945/35601$ destroys the period-3 limit cycle of the greedy algorithm. That this cycle is indeed realized is confirmed in the simulations, cf. Fig. 4. With the boundary at x_0, the interpretation of the new strategy is straighforward: it is a more intelligent greedy strategy, in which game A is now played whenever the expected gain of B is below $1/2 - (13/10)x_0$ (this result is obtained from eq. (5) with $\{s_{[0,1,0]}, s_{[0,1,0]}, s_{[0,1,0]}\} \to \{0, 0, 0\}$ (since game B is played) and $P_{[1,0,0]}(n) \to x_0$).

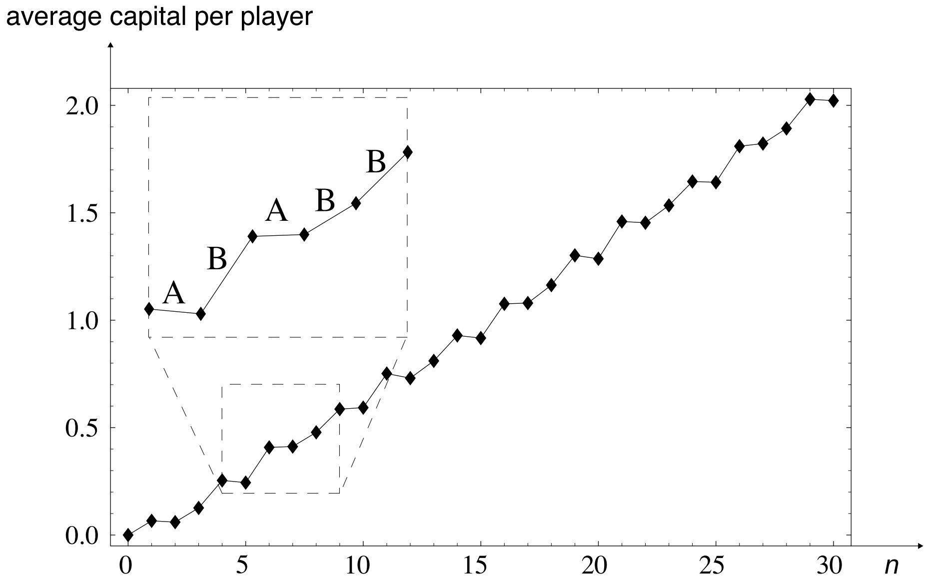


Figure 4. Time evolution of the average capital (per player) of a collection of 10000 players, using the more intelligent greedy strategy (with the boundary line at $x_0 = 0.334$). It is clear that the gain received by the players is periodic in time: after five timesteps, the same gain is obtained. This cycle can be identified by the $ABABB$ game sequence, as shown in the inset. Note that fluctuations in the gain are still significant for 10000 players.

5. DISCUSSION

We close with some final remarks. First it should be clear from the mean field analysis that the gain is usually changing in function of an adopted pure strategy in a abrupt way, since cycles appear or disappear when the constituting points move into or out of the appropriate region. For example, upon moving the decision boundary starting from the greedy strategy at $x_0 = 5/13$ to smaller values, the performance of the plain greedy algorithm will persist until the increased gain of the intelligent version is abruptly reached when x_0 becomes smaller than 11945/35601. Second, we note that non-periodic sequences, generated by various chaotic time series, were considered in Ref. 10. We have not discussed these sequences here because they do not seem to have a simple strategic interpretation, and require for their generation, as far as we can see, fractal boundaries between the A and B regions. Third, there is the possibility for coexisting attractors. In this case, the gain will depend on the initial condition. Such a situation occurs for example when the boundary line of the greedy strategy is shifted to the left to a value $x_0 \in]11945/35601, 3391159/9549529[$. In this case, both the period-3 and period-5 limit cycle can occur. Fourth, we were only able to answer the question of the optimal strategy for $N = 1, 2$ and 3. The identification of the optimal strategy for $N \to \infty$ is an open problem, including the question of whether or not it has a fractal nature. This difficult question may be answered by introducing further simplifications in the Parrondo game. A very promising primary model involving two rather than three states was recently introduced,[11] where such an analysis may indeed be performed.

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