Invited Paper

How much mathematics should optics students know ?

Anna Consortini

Physics Department, University of Florence
via S.Marta 3, 50139 Florence, Italy

Abstract

An attempt is made to answer the question of how much mathematics optics students should know, not only to understand optics, but also to be able to work with optics in the future. Fundamental optics and also some fields of application in optics or related disciplines are examined, in order to specify the mathematics required.

1. INTRODUCTION

By "optics students" I mean University undergraduates and graduates. Some of the initial concepts could also be applied to high school students.
Here we are concerned with the mathematics required for the foundations of classic and modern optics with an occasional look at quantum optics.
Two different levels of learning optics are considered. The first level includes the fundamental laws of optics and the minimum amount of mathematics needed to understand them. The second level includes a deeper insight into the laws and a number of modern applications that require a more profound knowledge of mathematics. Some applications and the required specialized mathematics are mentioned as well.

Initially optics meant the visible spectrum, but more recently infrared and ultra violet frequencies have also been included. We could use the term "optics" in a broad sense and not set any limitations to the frequencies, by observing that the laws of optics are applicable to all frequencies, provided that the appropriate scaling relations are satisfied. It should be remembered that one can talk of microwave optics and that radio waves are reflected by the ionosphere and diffracted by mountain tops. However, we prefer here to make reference to the term "optics" in its commonly accepted meaning.

A list of books on optics, some of which have already become classics, has been added to the end of this paper. This list is not intended to be exhaustive but only an indication of some examples. It includes books which require a considerable knowledge of mathematics on the part of the reader, some that are less demanding in this field, some which treat specific subjects and others dealing with modern applications. A good example of this last type is the very recent ICO volume "International Trends in Optics" where aspiring researchers can read about modern trends in optics and have an idea of the mathematics involved.

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2. GEOMETRICAL OPTICS

2.1. Homogeneous media

Geometrical optics, alternatively referred to as ray optics, represents the first approximation of wave propagation.

At a very first stage students may not know the conditions under which geometrical optics is valid, but, as soon as diffraction is introduced, this point must be clearly stated, by using the appropriate mathematics. Here it suffices to say that light propagates in free space or homogeneous media and encounters discontinuities represented by plane surfaces or by curved surfaces that in the region of interest can be "confused" with their tangent plane.

As the name clearly suggests, geometrical optics requires knowledge of geometry, more precisely classical elementary geometry. Reflection laws and behaviour of all instruments that are based on reflecting surfaces (such as mirrors) and also some simple examples of aberrations in these instruments can be understood by geometry and algebra. The idea of laser cavity (e.g. confocal) can also be given through simple reflection.

Trigonometry is required to suitably describe the laws of refraction and total reflection. Here the distinction between geometry and trigonometry is made for the purpose of distinguishing their different roles in geometrical optics.

Geometry and trigonometry allow one to understand the fundamentals of systems employing refracting surfaces (e.g. lenses), image formation in the paraxial approximation, and simple aberrations including chromatic ones.

The possibility of guiding radiation in plane or cylindrical homogeneous structures can also be understood by means of total reflection.

At this first level the study of the aberrations in both reflecting and refracting optical systems requires knowledge of power series expansion of simple functions, which in turn implies some knowledge of the theory of functions and of differential calculus.

2.2. Inhomogeneous media

Integral calculus is appropriate for ray propagation in inhomogeneous media, in particular for writing in explicit form Fermat's principle for continuous media or for continuous media where some discontinuities are present.

As is well known, according to Fermat's principle, light propagates from a point A to point B along the curve (or the curves) along which the transit time is an extremal that is it has a minimum, or a maximum, or a stationary value with respect to nearby paths. In the path integral

$$t = \frac{1}{c} \int_{\Gamma} n(P)\, dl \qquad (1)$$

where n(P) denotes refractive index, the unknown quantity is path Γ. Derivation of path Γ for a given function n(P) constitutes a variational problem, that can be solved with different methods, one of which is that of Lagrange's invariants.

At the lower level, however, one can skip this mathematics by simply getting the students to check the property in a number of cases. It is particularly simple

to check the property in the case of reflection and refraction laws. In addition, in my opinion, it is very instructive to check that the image (real or virtual) of a source point, given by a perfect system, is the cross point of rays whose transit times from source to image are equal. Plane mirror, or lenses in the paraxial approximation, or parabolic and elliptical mirrors, when the source is in the focus, are suitable examples for verifying that the transit time is stationary. This will be useful subsequently for understanding the process of image formation by means of phase considerations.
As to the higher level, in general students should know how to solve variational problems, and to derive the corresponding differential Euler equations: one equation or a system depending on the number of variables. They should also know the rules for solving differential equations, (if possible) and at least the solutions of a number of such equations. A simple example to give the student is that of a path in one plane, where the refractive index is a function of one variable in the plane. A case of this type arises in the study of meridional rays in graded index fibers. For those fibers the refractive index is a function of only the distance from the fiber axis and the plane of the path is a plane containing the fiber axis.

Clearly for those that will have to work with these types of problems adequate mathematical preparation is necessary for solving the differential equations of different types that can occur and sometimes for handling these equations by computer.
Apart from this last point, the mathematics required up to now is not considerable and is not different from the basic mathematics that a university student knows or learns at the very beginning. Therefore we can conclude that learning ray optics does not require a particular effort with mathematics. However, use of geometrical optics (ray optics) for modern investigations and specialized applications, such as propagation in non homogeneous media or in random media, requires greater mathematical knowledge.

3. WAVE OPTICS - OPTICAL APPROXIMATION

3.1 - Free propagation in vacuum and in homogeneous transparent media - Interference.

As soon as the wave nature of light is dealt with one is faced with much greater mathematical requirements.
As a first introductory step, the electromagnetic nature of light is generally outlined, even at a lower level. Differential and integral calculus, and vector calculus are necessary in the study of the fundamental laws of electromagnetism and in the development of Maxwell's equations. Partial derivatives are required to pass from the integral to the differential forms of Maxwell's equations in free space or homogeneous media and to obtain the wave equation for vectors. Although not strictly required at a lower level, at a higher level the student needs to see the different forms of the wave equations in different reference systems and their solutions, for future handling of waves of different types. Use of complex vectors to describe the two independent solutions of Maxwell's equations is appropriate for the higher level case. At the lower level, introduction of the complex form can be delayed until the optics approximation is introduced.
The optics approximation, when applicable, allows one to treat many optical

phenomena without vectors, by considering a Cartesian component v(P,t) of the field as representative of the entire field. This allows the student to understand many optics phenomena even at the lower level. At the lower level a short justification can be given for the introduction of the optics approximation, while at a higher level some more considerations based on vector analysis could be useful. At this stage, it is necessary for the students to know how to write the component v(P,t) in a complex form, and to understand the meaning of this use that greatly helps the subsequent development of the course. At the lower level this need not be strictly required, because, as is the case in some textbooks, trigonometry could be sufficient. In my opinion, however, in most cases it is unnecessary to burden students with a number of cumbersome trigonometric formulas, when amplitude and phase helps so much. Complex form allows one to introduce the time dependence of a monochromatic wave simply by means of a factor $\exp(\mp i\omega t)$ that constitutes a first step towards future wave trains, polychromatic signals and impulses. [Concerning the sign of imaginary unit "i" in the exponent, the minus sign is the standard recent use in optics, although some authors use the plus sign. It could also be useful to tell the students that they can find "-j" used in texts dealing with electromagnetic waves].
In addition, the complex form permits an easy separation between space and time variables

$$v(P,t) = u(P)\, e^{-i\omega t} \tag{2}$$

leaving a complete generality for the complex amplitude u(P). In turn u(P) can be easily specialized to represent the different wave forms; plane, spherical, cylindrical waves are described by their corresponding geometrical surfaces having constant phases. Evanescent waves, or leaky or dissociated waves, can be handled too. Gaussian beams, that derive their name from amplitude, and more precisely the fundamental modes emitted by lasers, can be easily described in terms of Gaussian distribution, a generally well known function even at the lower level. When beams are studied at a higher level, they require some more mathematics. Description of higher order modes needs Hermite polynomials. Hermite polynomials are also useful in the study of the behaviour of some basic laser cavities.

The complex form of the field is required for the presentation of the Huygens-Fresnel principle, at an elementary level and for describing interference patterns obtained by different kinds of waves that are produced by interferometers of different kinds. Describing interference (and later the interferometers' resolving power) some concepts of the mathematics of coherence are appropriate. Speaking of coherence some fundamentals of the probability theory and definitions of quantities like average and variance are important. Generally the theory of coherence is not known to students and the optics teacher should introduce the concept of time and space coherence, of partially coherent (scalar) fields, the definition of correlation and covariance (= correlation of fluctuations) functions of first order and the degree of coherence that is related to fringe visibility.Higher order correlations will be useful for studying the laser, at a higher level.

As the last point about free propagation one generally describes the derivation of the ray equation,and the "eiconal" where the students learn when diffraction can be neglected. By introducing amplitude and phase in the wave equation much instructive information is found, including that about diffraction taking place where there is an abrupt discontinuity in amplitude. The required mathematics involves some differential operators such as gradient, Laplacian ∇^2, and so on, already encountered before. The key point is that diffraction takes place where the term $\nabla^2 A$ is not negligible with respect to $An^2k_0^2$ where A denotes amplitude, k_0 wavenumber in the free space, and n refractive index. This implies that amplitude variations (second difference) taking place in the space of a wavelength must be negligible in order to neglect diffraction.

3.2- Effects of boundaries - Diffraction

There are several ways of introducing scalar diffraction to the students, without entering into the problems and details of a rigorous theory of electromagnetic fields as solutions of Maxwell's equations with the appropriate boundary conditions. Let us mention: 1) the Huygens-Fresnel principle, 2) the Helmholtz-Kirchhoff theory and formula and 3) the principle of inverse interference (Toraldo) or expansion in plane waves (Duffieux).

1) - Although in general Fresnel diffraction is described to students, much more attention is generally paid to Fraunhofer diffraction, because it is the most typical case in optics. Presentation of the Huygens-Fresnel principle does not require any additional mathematics and it is generally given in the simplest mathematical form without taking into account any angular dependence. Sometimes the angular factor $(1+\cos\theta)/2$ is used, were θ denotes the angle with respect to the direction of the impinging wave, as derived from the Helmholtz-Kirchhoff formula. Justification of the phase factor $\exp(i\pi/2)$ for the spherical wavelet, as well as the presence of wavelength in the amplitude is left to higher level stages.

2) - Kirchhoff's theory requires knowledge of the method of Green's function for the solution of differential equations but generally it can also be presented at the lower level to students who do not know the method. Green's method, however, will be necessary and very useful in the presence of sources.

Examples of diffraction can be given in a two dimensional space (plane); the typical example is diffraction by a thin slit infinitely extended orthogonally to the plane. Through the Huygens-Fresnel principle and the Kirchhoff formula, the Fraunhofer pattern of a plane wave incident normally to the slit is found to by given by a $\mathrm{Sinc}(x) = (\sin x)/x$ function, not difficult to be evaluated by students at all levels. Students should pay attention to the approximations involved. Diffraction at the infinity of a circular aperture gives rise to the Bessel function $J_1(x)/x$, a well known pattern, whose square is known as Airy pattern. In case the students are not already acquainted with Bessel functions, here it is necessary for them to learn them and their properties including their asymptotic behaviour, also because of the importance of the above function in the concept of resolving power of optical systems. Some considerations on the energy contained in the central disc (Airy disc, or main lobe) with respect to the total power can be made by integrating this function. By the way, let us also note that knowledge of the Bessel function of zero order can also allow one to introduce the so called "non-diffracting beams".

Some further comment based on the considerations made at the end of sect.

2, and also on the approximate evaluation of some integrals (see below) can help the student to realize that diffraction is present every time light suffers an abrupt discontinuity of amplitude. This takes place at the border of any aperture and not only when light crosses an aperture smaller or of the same order as the wavelength, (this error is present in many textbooks).
The above considerations are the basis of the ray theory of diffraction, developed by Keller et alii. The required mathematics, in the scalar case, is not different from the mathematics required up to now.
3) - The inverse interference principle allows one to present diffraction in a way that immediately visualizes the Fourier transform properties of diffraction and introduces students to Fourier optics. The inverse interference rule consists of developing the field diffracted by an aperture in an ensemble (complete including evanescent ones) of plane waves propagating away from the aperture in different directions and requiring that a) they reproduce the field over the aperture and b) the field vanishes at infinity according to Sommerfeld's condition of radiation. The validity of this procedure is guaranteed by the uniqueness of the solution.
For a periodic plane aperture the system reproduces the Fourier expansion of the field on the aperture and the space frequencies are related to the directions of the diffracted plane waves (orders), including evanescent waves. Students are required to know the development of periodic functions in Fourier series (bidimensional). This mathematics allows one to describe diffraction from gratings of any type including those on non planar surfaces.
For a non-periodic aperture the system obtained by using the interference rule consists of a continuum of plane waves (with respect to direction) that represents the decomposition of the field in a continuum of Fourier terms. The amplitude of the field diffracted at infinity is the Fourier transform of the field on the aperture. The above results of course can also be obtained by using the Kirchhoff theorem as it can be checked by the Fraunhofer diffraction pattern of slit and circular aperture.
By the way, we also note that this method allows one to easily explain the holographic procedure of wavefront reconstruction. The diffraction from apertures illuminated by partially coherent light can also be easily derived.

Some additional comments on the diffraction for students who will go on to research. It is important that they know that diffraction results are only approximate, that complete description is a problem solved only in a very few cases e.g diffraction by an infinite half plane (Sommerfeld). They should also know that the different approximate formulas, that have been almost always found to work very well, can sometimes fail when they are used at the limits of their validity. This happened to me in the investigation of open cavities having a low Fresnel number, for laser applications, that required a numerical iterative solution of an integral equation. I found that different forms of the Huygens-Fresnel principle, that differ in angular dependence, can give rise to different results some of which were unacceptable. For instance in very low-loss cavities "negative losses" were the results if the angular dependence was not taken into account, or if the dependence given by the Helmholtz-Kirchhoff formula $[1+\cos\theta]/2$ was used. Only the dependence $\cos\theta$ as given by the Rayleigh-Luneberg formula did not give negative losses.

3.3 - Images and aberrations

The theory of image formation can be given in terms of diffraction (by apertures) and interference of the waves emitted from the source after crossing an optical system, say a lens. The first step is the Abbe theory of image formation in the microscope, where no additional mathematics is required. By using diffraction it is possible to write the image of an object, given by a perfect system, as a product (convolution) of the "spread function" or image of a source point and of the distribution of complex amplitude on the object for the coherent case (intensity for the incoherent case). At this point it is not indispensable for students to know convolution. However, if they already know it, it is appropriate that the relative terminology be used.
A diffraction theory of aberrations has also been developed. To describe aberrations, knowledge of the circle polynomials of Zernike is required.

4 - FOURIER OPTICS - IMAGES - HOLOGRAPHY - IMAGE PROCESSING

As is clear from the name, Fourier optics requires an adequate knowledge of Fourier Transforms, a subject that can be easy handled in the case of Engineering and Physics students and is less easy for students of other faculties (e.g. medicine, biology and so on). By adequate knowledge we mean that students need not only to know the definitions, but to become familiar with a number of properties and theorems (such as sampling theorem, convolution etc..) that are the basis for many practical applications. The ideal preparation on this subject should include the theory of complex variable functions and the theory of distributions (or generalized functions) that allow the student to handle any type of mathematical description of optical signals without trouble about or doubts on the validity of the results. Terms like "spread function" "optical transfer function" "modulation transfer function" simultaneously correspond to precise mathematical operations and optical entities. It is hard to exaggerate the importance for students to have a clear mathematical understanding of them in order to be able to handle optics in the laboratory. A number of typical Fourier transforms that are commonly used can be derived and understood by the students. In addition, they are required to know a number of related theorems and to be able to see their practical application. Convolution, and the relation between the spectra in a convolution are basic for understanding the image formation process, the limits in the resolving power of optical systems, the holography and the methods of optical information processing, such as filtering or image enhancement.

5 - OPTICAL FIELDS - BOUNDARIES - ANYSOTROPIC MEDIA - GUIDED PROPAGATION - SCATTERING

Scalar approximation does not always work in optics. There are a number of important cases where the vector nature of the field in optics needs to be taken into account.

One case is the interaction of light with boundaries between two conducting or nonconducting media where polarization plays an important role. Another case is propagation in anysotropic media. These two subjects involve many important physical properties that give rise to important results and applications. We will not enter into details because, apart from the difficulty of dealing with vectors and the use of matrix or tensor formalism, no special mathematics is required.

As to guided propagation, that in planar step index guides does not involve special mathematical difficulties, however it requires knowledge of the concept of modes. For graded index planar guides one useful mathematical method is WKB. Guided propagation in dielectric fibers requires additional mathematics. Propagation in dielectric fibers is generally treated by writing Maxwell equations in a cylindrical system of coordinates. Even in the simple case of a step index guide the radius dependent factors of the solutions (modes) are found in terms of Bessel or Hankel functions of both real and imaginary argument, and the conditions at the boundaries involve these functions. Knowledge of these functions and their recurrence formulas, their series development, as well as their asymptotic behaviour and the relationship between different independent couples of them is necessary.

In the case of graded index fibers the WKB method can be used with careful attention to be paid to turning points where the Airy function formalism is involved.

The vectorial nature of the field cannot be neglected in scattering from particles, either small or large non absorbing ones. For small spherical particles there are no special mathematical requirements, but as soon as scattering from larger particles is considered (Mie scattering) one has to deal with vectors and with Legendre polynomials, associated Legendre functions and spherical Bessel functions.

Propagation through scattering media is a problem of a statistical nature (see statistical optics). The case of very low density media can be treated by considering single scattering, while for more dense media one has to deal with problems of multiple scattering.

Another subject where polarization must be taken into account is scattering from rough surfaces, that is now a subject of active research, with particular interest in enhanced back scattering.

6 - NON MONOCHROMATIC SIGNALS - COHERENCE

Knowledge of Fourier transforms also allows one to account for non monochromatic signals by simply expressing a signal as the addition (integral) of elementary components. Handling non monochromatic signals can be then reduced to the study of their Fourier components.

No particular new mathematical requirement is necessary to find, for instance, the group velocity in a dispersive medium. Foundations of Laplace transforms could help in finding the velocity of an optical signal.

Dealing with non monochromatic optical radiation, foundations of time coherence theory are required, as already mentioned. The concept of analytic signal, of correlation function and spectral density as well as the Wiener-Khintchine theorem are necessary.

The coherence matrix is also useful for describing Stokes parameters.

7 - STATISTICAL OPTICS - COHERENCE - PROPAGATION AND SCATTERING IN RANDOM MEDIA.

As is clear form the name, statistical optics involves all those topics, including noise, that require a statistical approach. Without entering into the details of this large field, here we will limit ourselves to mentioning some subjects and the mathematics they require. A common basis is, of course, knowledge of fundamental statistics: a number of different probability distributions, definition of averages, of moments centered or not and some useful parameters, like skewness or excess, and so on. The theory of coherence, (time and space coherence) is required as well as correlation or covariance functions of first and higher orders, and power spectral densities. The description of light emission by incoherent (e.g.thermal) and coherent (laser) sources can be made in terms of the probability distribution of intensity and photon counting statistics, respectively. The effect of noise in receiving systems as well as in the photographic process involves statistics. Speckle patterns produced by laser light follows a negative exponential probability density function for intensity. A number of probability density functions (pdf) have been used for describing intensity fluctuations of coherent light propagated through random media. A typical example is turbulence in clear atmosphere where a number of different distributions can be found, depending on the strength of the turbulence and evolving from the lognormal pdf for small fluctuations to the negative exponential in the limit of very strong fluctuations.

The study of propagation in random media gives rise to a parabolic equation for the correlation functions of the complex amplitude or the moments of the scalar field. That equation has been solved only for the second order moment, while the fourth order moment equation, of interest for intensity fluctuations, has been treated approximately.

Another way of treating statistical problems of propagation is numerically, through Monte Carlo methods of simulation. The methods are useful for both propagation through clear atmosphere and for scattering, in particular multiple scattering in dense media.

A description of the average behaviour of optical systems in random media, for instance the locally stationary atmosphere, can be given in terms of MTF of the atmosphere. This in turn requires knowledge of the so-called wave structure function, which can be obtained from the equation of the second order correlation of amplitude. For a reasonable model of atmospheric turbulence (based on the Kolmogorov spectrum), the solution is found in terms of hypergeometric functions. Knowledge of these functions and of a number of their properties is required. Integrals involving these functions are sometimes necessary.

It is to be noted that hypergeometric functions are found very often in solving problems of propagation in random media.

8 - SPECIAL MATHEMATICS FOR OPTICS

There are a number of functions, of integrals or of methods to evaluate integrals, that are typical of optics and that are important for both learning and doing research. Some examples follow.

1 - The stationary phase method permits the evaluation of integrals where, in addition to a slowly varying function, there is a rapidly varying function that depends on a parameter whose value is high. A typical example is a phase factor like exp(iks) where wavenumber $k=2\pi/\lambda$ has a large value. The stationary phase method allows one to recognize the point from where the maximum contribution to the integral arises and to give a series development in powers of 1/k rapidly decreasing. By applying this method one can evaluate the group velocity in a

dispersive medium and derive the theorem of the total elastic scattering cross-section, well known as "optical theorem" in quantum mechanics.
2 - The residuals theorem belongs to the theory of complex variable functions and allows one to easily evaluate a number of integrals of interest in optics such as the "complete Fresnel integral", the integral of (exp(ix))/x and the Fourier transform of a Gaussian function.
3 - The steepest descent path method also belongs to the theory of complex variable functions and, as an alternative to the stationary phase method, permits the asymptotic evaluation of integrals and is here of particular interest because by using it one can evaluate the asymptotic development of the Airy functions Ai(z) and Bi(z)

9 - CONCLUSIONS

For those who plan to do research in optics there are many specialized fields where additional mathematics is required.
In some cases the same mathematics can be applied to different problems. For example being able to handle problems of eigenvalues and eigenfunctions is necessary in the study of modes and losses of laser cavities, and in propagation through fibers, as already mentioned. In addition, it is necessary in describing the degrees of freedom of images, a more modern measure of the resolving power and a basic quantity in optical information theory, basic to understand the practical limit set by noise to superresolution. Eigenfunctions can also be important for a number of applications including adaptive optical systems.
Working with instability and chaos problems as well as in non linear optics involves the solution of non linear differential equations and so on.

As appears from this short and inevitably incomplete analysis, much mathematics is required even for the low level learning of optics. However this mathematics is not more than that an ordinary student in scientific faculties learns. Much more knowledge of mathematics is required at the higher level, especially for those who wish to work with optics in specialized fields.

However, it is to be pointed out that it is not absolutely necessary that the students remember the demonstrations of all theorems, for example those of existence and uniqueness. However they need to know the necessary and sufficient conditions under which certain operations are valid. They need to know very well how to use them and within what limits. They need to know a number of special functions, and also the behaviour of these functions in limiting cases, such as series power development for small values of the variable, asymptotic series, relations between functions of different order, integrals involving functions and so on. Fundamental at this point is that they become acquainted with some basic books of tables and functions, where most definitions can be found together with limits and conditions of validity and so on. And mostly they need practice to remove mistaken ideas about the difficulty of using mathematics. Of course a good computer software, if available, can be of great help for numerical evaluations in particular cases.

Those that are going to work in optics research do not need more or less mathematics than researchers in other fields. May be in optics, like electromagnetism in general and more than in some other fields,

experimentalists also need to know mathematics, because very often an experiment or an experimental step corresponds to a precise mathematical operation.
At this point another question arises: how much mathematics and at what level should optics teachers know?

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